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On Eigenvalues of Complement Digraphs

Prayitno, M. I. A.* [1], Susanti, Y. [1], Wahyuni, S.* [1], and Suparwanto, A. [1]

[1] *Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Gadjah Mada, Sekip Utara Bulaksumur, Yogyakarta, Indonesia, 55281*

E-mail: swahyuni@ugm.ac.id
**Corresponding author*

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Abstract

A digraph's antiadjacency matrix is defined as its complement's adjacency matrix. Therefore, we can distinguish the complement of digraphs by analysing the properties of their antiadjacency matrices. In this paper, our interest lies in exploring the properties of eigenvalues of the antiadjacency matrix of digraphs and establishing their relation to the characterisation of digraphs. Recent results regarding the eigenvalues of the antiadjacency matrices of certain classes of cyclic digraphs allow us to generalise the bounds of the spectral radius of a complement digraph. Additionally, we establish a connection between the bounds of the spectral radius of a complement digraph and the characterisation of the complement digraph. Since a digraph can be either cyclic or acyclic, we distinguish between the spectral radius of a cyclic and acyclic digraphs.

Keywords: acyclic digraph; adjacency matrix; antiadjacency matrix; complement digraphs; cyclic digraph; eigenvalues; spectral radius.

1 Introduction

Recent studies have revealed various properties of the spectral radius of a digraph. The existence of a real number as the spectral radius and the relationship between the eigenvalues of a digraph and its characterisations are discussed [17]. The bounds for the spectral radius of a digraph are established [16]. Therefore, we are intrigued by the discovery of the disparities between the eigenvalues of digraphs and their complements. Given that a digraph can contain directed cycles as subdigraphs, it can be cyclic or acyclic. Thus, our paper delves into the characteristics of both cyclic and acyclic digraphs.

Some results on eigenvalues of the complement of an undirected graph have been discussed, such as in [11]. As mentioned in Section 2 in [11], the eigenvalues of the adjacency matrix of the complement graph of the hypercube graph have been computed. Moreover, the properties of the complement of the undirected graph have been discussed in [5]. Unlike the study in [11, 5], the authors are focussing on the properties of the complement of a graph of several classes, including the cycle graph. Therefore, we are motivated to find the characteristics of the complement of digraphs through their adjacency matrix.

The concept of a digraph antiadjacency matrix of a digraph was introduced [3]. According to its definition, a digraph's antiadjacency matrix is equal to the adjacency matrix of its complement. Furthermore, a digraph's adjacency matrix is equal to the antiadjacency matrix of its complement. Thus, by analysing the properties of the antiadjacency matrix of a digraph, we can distinguish the properties of a digraph and its complement. Therefore, our aim is to obtain the eigenvalues of the antiadjacency matrix to infer the eigenvalues of the adjacency matrix of its complement. Furthermore, we will elucidate some of the relationships between the eigenvalues of adjacency and antiadjacency matrices and the characterisation of digraphs.

The eigenvalues of the complement of the directed-unicyclic flower vase graph have been studied in [1]. Meanwhile, the properties of the eigenvalues of the complement of the directed-unicyclic corona graph have also been discussed [12]. The results on the eigenvalues of the complement of the directed cyclic sun graph have been obtained [13]. Furthermore, in [15], we have several properties of the eigenvalues of the complement of the directed-cyclic wheel graph. Through our analysis, we have identified a discernible pattern in the eigenvalues of the complement digraph of a cyclic digraph, particularly in terms of its spectral radius. Consequently, our aim is to demonstrate the relationship between the spectral radius of the complement of a cyclic digraph and its cycle subdigraph.

We are also interested in determining the spectral radius for an acyclic digraph case. Initially, we focused on obtaining the spectral radius of the complement of a path digraph. Through our analysis, we have managed to derive the characteristic polynomial of the complement of a path digraph, which we find to be an arithmetico-geometric series. However, our current capability is limited to establishing the bounds of the spectral radius of the complement of a path digraph. Moreover, our investigation aims to identify the bounds of the spectral radius of the complement of an acyclic digraph.

2 Preliminaries

All the digraphs (directed graphs) discussed herein are considered unoriented, potentially containing directed loops, and are not multidigraphs. In this paper, we refer to the definition of a digraph from [6]. For terminology related to oriented digraphs, subdigraphs, as well as set in-neighborhoods and out-neighborhoods, we refer to [7]. The definitions for directed walks, directed paths, and directed cycles are based on [8]. Furthermore, for the definition and terminologies related to path digraphs and cycle digraphs, we refer to [10].

2.1 Directed graphs

A digraph (directed graph) $\overrightarrow{\Delta}$ is an ordered pair of $\left(V_{\overrightarrow{\Delta}}, A_{\overrightarrow{\Delta}}\right)$, where $V_{\overrightarrow{\Delta}}$ is a vertex set of $\overrightarrow{\Delta}$ and $A_{\overrightarrow{\Delta}}$ is an arc set of $\overrightarrow{\Delta}$ that are disjoint from $V_{\overrightarrow{\Delta}}$, along with incident functions $\psi_{\overrightarrow{\Delta}}$ associating every arc of $\overrightarrow{\Delta}$ with an ordered pair of vertices of $\overrightarrow{\Delta}$. In this paper, if $\upsilon, \nu \in V_{\overrightarrow{\Delta}}$ and $\alpha \in A_{\overrightarrow{\Delta}}$ with $\psi_{\overrightarrow{\Delta}}(\alpha) = (\upsilon, \nu)$, the arc α is denoted by $\alpha = (\upsilon, \nu)$ for simplification.

An arc α is said to join υ to ν if and only if $\alpha = (\upsilon, \nu) \in A_{\overrightarrow{\Delta}}$. The order of $\overrightarrow{\Delta}$ is defined as $|V\left(\overrightarrow{\Delta}\right)|$, while the size of $\overrightarrow{\Delta}$ is defined as $|A\left(\overrightarrow{\Delta}\right)|$.

A digraph $\overrightarrow{\Delta}$ is said to be oriented if for every $\upsilon, \nu \in V_{\overrightarrow{\Delta}}$, $(\upsilon, \nu) \in A_{\overrightarrow{\Delta}}$ implies $(\nu, \upsilon) \notin A_{\overrightarrow{\Delta}}$. A directed loop is an arc that joins a vertex to itself. Some arcs of $\overrightarrow{\Delta}$ are said to be parallel if those arcs are joining the same vertices in the same direction. A digraph is called a multidigraph if and only if there exist parallel arcs. We say that a digraph $\overrightarrow{\Delta}$ is simple if it does not contain any directed loops or parallel arcs. The digraphs $\overrightarrow{\Delta}$ and $\overrightarrow{H}$ are said to be isomorphic if there exists a bijective function $\phi : V_{\overrightarrow{\Delta}} \to V_{\overrightarrow{H}}$ such that for any $\upsilon, \nu \in V_{\overrightarrow{\Delta}}$, $(\upsilon, \nu) \in A_{\overrightarrow{\Delta}}$ if and only if $(\phi(\upsilon), \phi(\nu)) \in A_{\overrightarrow{H}}$.

For every arc $a = (\upsilon, \nu) \in V_{\overrightarrow{\Delta}}$, we say that the vertex υ is adjacent to the vertex ν, while the vertex ν is said to be adjacent from the vertex υ. The out-neighbourhood set of the vertex $x \in V_{\overrightarrow{\Delta}}$ is defined as $N^+(x) = \left\{y : (x, y) \in A_{\overrightarrow{\Delta}}\right\}$, while the in-neighbourhood set of the vertex $x \in V_{\overrightarrow{\Delta}}$ is defined as $N^-(x)\left\{y : (y, x) \in A_{\overrightarrow{\Delta}}\right\}$. Furthermore, normally, we have the in-degree is defined as $id_{\overrightarrow{\Delta}}(x) = |N^-(x)|$, while the out-degree of the vertex $x \in V_{\overrightarrow{\Delta}}$ is defined as $od_{\overrightarrow{\Delta}}(x) = |N^+(x)|$.

A digraph $\overrightarrow{H}$ is called a subdigraph of a digraph $\overrightarrow{\Delta}$ if $V_{\overrightarrow{H}} \subseteq V_{\overrightarrow{\Delta}}$ and $A_{\overrightarrow{H}} \subseteq A_{\overrightarrow{\Delta}}$. The subdigraph $\overrightarrow{H}$ is called the induced subdigraph of $\overrightarrow{\Delta}$ whenever for every $\upsilon, \nu \in V_{\overrightarrow{H}}$, $(\upsilon, \nu) \in A_{\overrightarrow{\Delta}}$ implies $(\upsilon, \nu) \in A_{\overrightarrow{H}}$.

Let $\overrightarrow{\Delta}$ be a digraph with a set of vertices $V_{\overrightarrow{\Delta}}$ and a set of arcs $A_{\overrightarrow{\Delta}}$. By the complement of a digraph $\overrightarrow{\Delta}$, denoted by $\overline{\overrightarrow{\Delta}}$, we mean a digraph with a set of vertices $V_{\overline{\overrightarrow{\Delta}}} = V_{\overrightarrow{\Delta}}$ and a set of arcs $A_{\overline{\overrightarrow{\Delta}}} = \left(V_{\overrightarrow{\Delta}} \times V_{\overrightarrow{\Delta}}\right) \setminus A_{\overrightarrow{\Delta}}$. An example illustrating a digraph and its complement is given in Figure 1 below.

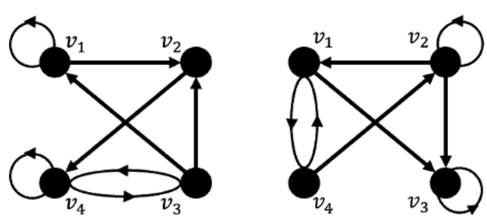

Figure 1: A digraph $\overrightarrow{\Delta_1}$ and its complement $\overline{\overrightarrow{\Delta_1}}$.

Given a digraph $\overrightarrow{\Delta}$ of order m and $\upsilon_i \in V_{\overrightarrow{\Delta}}$, for $0 \leq i \leq k$. We say that a sequence of vertices $\overrightarrow{W} = (\upsilon = \upsilon_0, \upsilon_1, \ldots, \upsilon_k = \nu)$, with $(\upsilon_i, \upsilon_{i+1}) \in A_{\overrightarrow{\Delta}}$ for $0 \leq i \leq k-1$, is a directed walk $\upsilon - \nu$ in digraph $\overrightarrow{\Delta}$. A directed walk $\upsilon - \nu$ is said to be closed if $\upsilon = \nu$, otherwise it is open. The number of arcs traversed by a directed walk $\upsilon - \nu$ is called the length of a directed walk-$\upsilon - \nu$. An open directed walk in a digraph $\overrightarrow{\Delta}$ where every vertex is traversed only once is called a directed path.

A directed path in a digraph $\overrightarrow{\Delta}$ that traverses every vertex in $\overrightarrow{\Delta}$ is called a directed Hamiltonian path in $\overrightarrow{\Delta}$. A directed cycle is a closed-directed path. A directed digon is a directed cycle of length 2. A digraph $\overrightarrow{\Delta}$ is said to be cyclic if there is a directed cycle; otherwise, it is called an acyclic digraph. We denote an acyclic digraph by $\overrightarrow{\Delta_a}$, while we denote a cyclic digraph by $\overrightarrow{\Delta_c}$. An example of both types of digraph is shown in Figure 2 below.

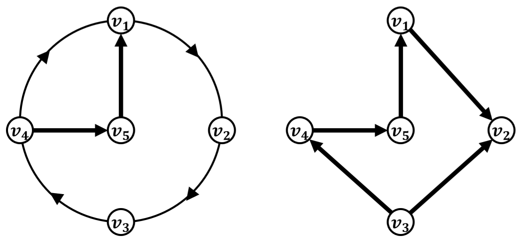

Figure 2: Cyclic digraph and acyclic digraph.

By a path digraph $\overrightarrow{\Pi_\eta}$, we mean a digraph that is isomorphic to a digraph that has a vertex set $\{\upsilon_i : i \leq \eta\}$ and an arc set $\{(\upsilon_i, \upsilon_{i+1}) : i \leq \eta - 1\}$. This implies that the complement of the path digraph is $\overline{\overrightarrow{\Pi_\eta}}$, which is isomorphic to a digraph that has a set of vertices $\{\upsilon_i : i \leq \eta$ and a set of arcs,

$$\{(\upsilon_i, \upsilon_j) : i, j \leq \eta\} \setminus \{(\upsilon_i, \upsilon_{i+1}) : i \leq \eta - 1\}.$$

An illustration of the directed path $\overrightarrow{\Pi_4}$ and its compliment $\overrightarrow{\Pi_4}$ is given in Figure 3 below.

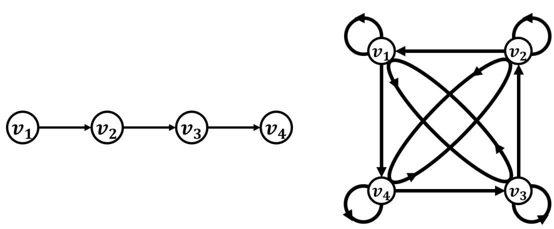

Figure 3: Directed path $\overrightarrow{\Pi_4}$ and its complement $\overline{\overrightarrow{\Pi_4}}$.

A cycle digraph $\overrightarrow{\Omega_\eta}$ is defined as a digraph that is isomorphic to a digraph with a vertex set $\{\upsilon_i : i \leq \eta\}$ and an arc set $\{(\upsilon_i, \upsilon_{i+1}); (\upsilon_\eta, \upsilon_1) : i \leq \eta\}$. According to the definition of cycle digraph and complement digraph, we know that the complement digraphs is a digraph that is isomorphic to a digraph with a vertex set $\{\upsilon_i : i \leq \eta\}$ and an arc set,

$$\{(\upsilon_i, \upsilon_j) : i, j \leq \eta\} \setminus \{(\upsilon_i, \upsilon_{i+1}); (\upsilon_\eta, \upsilon_1) : i \leq \eta\}.$$

An instance of a cycle digraph and its complement is displayed in Figure 4 below.

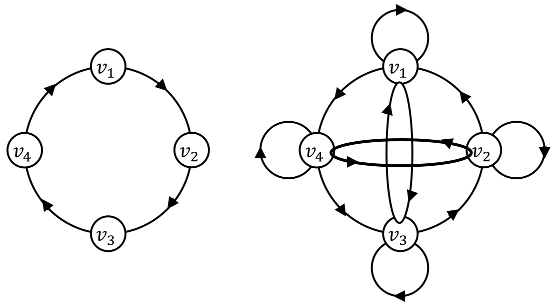

Figure 4: A cycle digraph $\overrightarrow{\Omega_4}$ and its complement $\overrightarrow{\overline{\Omega_4}}$.

2.2 Characteristic polynomial and eigenvalues

Given a spectrum $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ of an $n \times n$ matrix, where $k \leq n$. The spectral radius of A is defined as $\rho(A) = \max\{|\lambda_i| : \lambda_i \in \sigma(A)\}$.

Note that,

$$\alpha + (\alpha+\delta)\rho + (\alpha+2\delta)\rho^2 + \ldots + (\alpha+(m-1)\delta)\rho^{m-1} = \sum_{k=0}^{m-1}(\alpha+k\delta)\rho^k. \tag{1}$$

is an arithmetico-geometric series [14]. In this paper, we will show that the characteristic polynomial of $\overrightarrow{\overline{\Pi}}_\eta$, $B_{\overrightarrow{\overline{\Pi}}_\eta}$, is equivalent to the arithmetico-geometric series. Consequently, we show a property of the arithmetico-geometric series in the Theorem 2.2.1 below.

Theorem 2.2.1. [14] *The arithmetico-geometric series* $\sum_{k=0}^{m-1}(\alpha+k\delta)\rho^k$ *is equal to the following form:*

$$\frac{\alpha\left(1-\rho^n\right)}{1-\rho} - \frac{(m-1)\,\delta\rho^m}{1-\rho} + \frac{\rho\delta(1-\rho^{m-1})}{(1-\rho)^2}. \tag{2}$$

Any matrix $A = (a_{i,j})$ of size $m \times n$ is called a nonnegative matrix (denoted by $A \geq 0$) if $a_{i,j} \geq 0$ for every $i \leq m$ and $j \leq n$. If $a_{i,j} > 0$ for all $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$, then A is considered a positive matrix (denoted as $A > 0$) [2].

Theorem 2.2.2 below gives the relation of a matrix's principal submatrices and lower bound of its spectral radius.

Theorem 2.2.2. [4] *Given an $\eta \times \eta$ nonnegative matrix A. The spectral radius of any of principal submatrices of A is less or equal to $\rho(A)$.*

Theorem 2.2.3 below gives the lower and upper bounds of spectral radius of a nonnegative matrix.

Theorem 2.2.3. [4] *Given an $\eta \times \eta$ nonnegative matrix A. Let s_i for $i \leq \eta$ denote the sum of elements in the i th row of A, and let $\rho(A)$ be the spectral radius of A. Then,*

$$\min\{s_i | i \leq \eta\} \leq \rho(A) \leq \max\{s_i | i \leq \eta\}. \tag{3}$$

2.3 Adjacency and antiadjacency matrices

Adjacency and antiadjacency matrices can represent a digraph. The concepts and terms associated with these matrices for a digraph in this paper are adopted from [3].

By an adjacency matrix of a digraph $\overrightarrow{\Delta}$ of order η, we mean an $\eta \times \eta$ matrix $A_{\overrightarrow{\Delta}} = (\alpha_{ij})$ where $\alpha_{ij} = 1$ if there is an arc (υ_i, υ_j) in $\overrightarrow{\Delta}$ and $\alpha_{ij} = 0$ otherwise. Meanwhile, the antiadjacency matrix of $\overrightarrow{\Delta}$ is defined as an $\eta \times \eta$ matrix $B_{\overrightarrow{\Delta}} = (\beta_{ij})$ where $\beta_{ij} = 0$ if there is an arc (υ_i, υ_j) in $\overrightarrow{\Delta}$ and $\beta_{ij} = 1$ elsewhere. Thus, we know that $B_{\overrightarrow{\Delta}} = A_{\overline{\overrightarrow{\Delta}}}$ and $B_{\overline{\overrightarrow{\Delta}}} = A_{\overrightarrow{\Delta}}$. Consider the following lemma that shows that the necessary and sufficient condition of a digraph $\overrightarrow{\Delta}$ is cyclic.

Lemma 2.3.1. [17] *Given a digraph $\overrightarrow{\Delta}$ and $\sigma\left(A_{\overrightarrow{\Delta}}\right)$ be the spectrum of adjacency matrix of $\overrightarrow{\Delta}$. Then, there exists $\lambda \in \sigma\left(A_{\overrightarrow{\Delta}}\right)$ such that $\lambda \neq 0$ if and only if $\overrightarrow{\Delta}$ is cyclic.*

Given a digraph $\overrightarrow{\Delta}$ and its complement $\overline{\overrightarrow{\Delta}}$. Since a complement of a digraph is also a digraph, then there exists one of the eigenvalues of $A_{\overline{\overrightarrow{\Delta}}}$ is not zero if and only if $\overline{\overrightarrow{\Delta}}$ is cyclic.

Remark 2.3.1. *Given a digraph $\overrightarrow{\Delta}$ and its complement $\overline{\overrightarrow{\Delta}}$. Let $\sigma\left(A_{\overline{\overrightarrow{\Delta}}}\right)$ be the spectrum of the adjacency matrix of $\overline{\overrightarrow{\Delta}}$. Then, there exists $\lambda \in \sigma\left(A_{\overline{\overrightarrow{\Delta}}}\right)$ such that $\lambda \neq 0$ if and only if $\overline{\overrightarrow{\Delta}}$ is cyclic.*

Motivated by the lemma 2.3.1, in this paper we present the properties of the eigenvalues of $A_{\overline{\overrightarrow{\Delta}}}$.

Recall that a digraph $\overrightarrow{\Delta}$ is said to be strongly connected if for every $\upsilon_i, \upsilon_j \in V_{\overrightarrow{\Delta}}$ there exists a path $\upsilon_i - \upsilon_j$. The following theorem shows the bounds of $\rho\left(A_{\overrightarrow{\Delta}}\right)$.

Theorem 2.3.1. [16]. *Let $\overrightarrow{\Delta}$ be a strongly connected digraph and $A_{\overrightarrow{\Delta}}$ be its adjacency matrix. Then,*

$$\min\left\{od_{\overrightarrow{\Delta}}(\upsilon_i)\ :\ \upsilon_i \in V_{\overrightarrow{\Delta}}\right\} \leq \rho\left(A_{\overrightarrow{\Delta}}\right) \leq \max\{od_{\overrightarrow{\Delta}}(\upsilon_i)\ :\ \upsilon_i \in V_{\overrightarrow{\Delta}}\}. \tag{4}$$

The lower and upper bounds of the spectral radius of the adjacency matrix of a strongly connected digraph are known according to Theorem 2.3.1. Finding the lower and upper bounds of the antiadjacency matrix digraphs, or the spectral radius of the adjacency matrix of complement digraphs, is another task that interests us. This happens when one is aware of the distinction between the adjacency matrix of digraphs and their complement's lower and upper bounds on the spectral radius. In the following theorem, we present the determinant of $B_{\overrightarrow{\Delta_a}}$.

Theorem 2.3.2. [3] *If $\overrightarrow{\Delta_a}$ has a Hamiltonian path, then $\det\left(B_{\overrightarrow{\Delta_a}}\right) = 1$; and $\det\left(B_{\overrightarrow{\Delta_a}}\right) = 0$ otherwise.*

We know that a cycle digraph $\overrightarrow{\Omega_\eta}$ itself is a cyclic digraph. Therefore, firstly we investigate $\rho\left(B_{\overrightarrow{\Omega_\eta}}\right)$, which is equal to $\eta - 1$. From the results presented in [15], we obtain $\rho\left(B_{\overrightarrow{W_k}}\right) = \eta - 1$, where η is the order of the cycle in $\overrightarrow{W_k}$.

All the eigenvalues of the antiadjacency matrix of the directed unicyclic flower vase graph $\overrightarrow{C_nS_m}$ have been discussed in [1]. From these results, we observe that $\rho\left(B_{\overrightarrow{C_nS_m}}\right) = \eta - 1$, where

η is the order of the cycle in $\overrightarrow{C_nS_m}$. On the other hand, the eigenvalues of the antiadjacency matrix of the directed cyclic sun graph $\overrightarrow{C_n \circ \overline{K_r}}$ have been studied [13], from which we establish $\rho\left(B_{\overrightarrow{C_nS_m}}\right) = \eta - 1$, where η is the order of the cycle in $\overrightarrow{C_n \circ \overline{K_1}}$. Similarly, for the directed unicyclic corona graph $\overrightarrow{C_n \circ \overline{K_r}}$, the case has been established in [12].

Furthermore, we obtain $\rho\left(B_{\overrightarrow{C_nS_m}}\right) = \eta - 1$, where η is the order of the cycle in $\overrightarrow{C_n \circ \overline{K_r}}$. Those cyclic digraphs have only one cycle sub-digraph. Hence, we obtain a pattern of the lower bound of $\rho\left(B_{\overrightarrow{\Delta_c}}\right)$, that is, $\eta - 1$ where η is the order of the cycle in the cyclic digraph. Motivated by these results, we investigate the lower bound of $\rho\left(B_{\overrightarrow{\Delta_c}}\right)$. Furthermore, we also motivated to find the upper bound of $\rho\left(B_{\overrightarrow{\Delta_c}}\right)$. After discussing the bounds of $\rho\left(B_{\overrightarrow{\Delta_c}}\right)$, we are eager to find the bounds of $\rho\left(B_{\overrightarrow{\Delta_a}}\right)$.

To determine the spectral radius of $B_{\overrightarrow{\Delta_c}}$, we need certain properties of this matrix. One key property is given by a theorem below that reveals the determinant of $B_{\overrightarrow{\Omega_\eta}}$.

Theorem 2.3.3. [9] *Given a cycle digraph* $\overrightarrow{\Omega_\eta}$. *Then we have* $\det\left(B_{\overrightarrow{\Omega_\eta}}\right) = \eta - 1$.

According to the definition of the antiadjacency matrix, the antiadjacency matrix of digraphs must be nonnegative. Hence, we can use Theorems 2.2.2 and 2.2.3 to find the bounds of $\rho\left(B_{\overrightarrow{\Delta_c}}\right)$ and $\rho\left(B_{\overrightarrow{\Delta_a}}\right)$.

3 Eigenvalues of Complement Graphs

From our observation, finding all eigenvalues of $B_{\overrightarrow{\Delta}}$ is not feasible. However, we can find the relationship between the eigenvalues of $B_{\overrightarrow{\Delta}}$ and the characterizations of $\overrightarrow{\Delta}$. Some of the relations are stated in the theorem below.

Theorem 3.0.1. *Consider a digraph* $\overrightarrow{\Delta}$ *of order* η. *Then,* $0 \in \sigma\left(B_{\overrightarrow{\Delta}}\right)$ *if and only if at least one of the following condition is satisfied:*

(*a*) *There exist* $\nu_i, \nu_j \in V_{\overrightarrow{\Delta}}$ *such that* $N^+(\nu_i) = N^+(\nu_j)$.

(*b*) *There exist* $\nu_i, \nu_j \in V_{\overrightarrow{\Delta}}$ *such that* $N^-(\nu_i) = N^-(\nu_j)$.

(*c*) *There exist* $\nu_i \in V_{\overrightarrow{\Delta}}$ *such that* $id_{\overrightarrow{\Delta}}(\nu_i) = \eta$.

(*d*) *There exist* $\nu_i \in V_{\overrightarrow{\Delta}}$ *such that* $od_{\overrightarrow{\Delta}}(\nu_i) = \eta$.

Proof. Let λ_i be the eigenvalues of B_Δ, $i = 1, 2, \ldots, n \leq \eta$. Suppose that one of the eigenvalues of $B_{\overrightarrow{\Delta}}$ is 0, then $\det\left(B_{\overrightarrow{\Delta}}\right) = \prod_{i=1}^{n} \lambda_i = 0$. According to the definition of the antiadjacency, $B_{\overrightarrow{\Delta}}$ has proportional rows and proportional columns if and only if there exist $\nu_i, \nu_j \in V_{\overrightarrow{\Delta}}$ such that $N^+(\nu_i) = N^+(\nu_j)$ or $N^-(\nu_i) = N^-(\nu_j)$. Moreover, $B_{\overrightarrow{\Delta}}$ has zero row(s) or zero column(s) if and only if $id_{\overrightarrow{\Delta}}(\nu_i) = n$ or $od_{\overrightarrow{\Delta}}(\nu_i) = n$. Hence, if one of the eigenvalues of $B_{\overrightarrow{\Delta}}$ is zero, then at least one of the conditions (a), (b), (c), or (d) is satisfied. Conversely, suppose that one of the condition (a), (b), (c), and (d) is satisfied. Therefore, there are proportional rows, proportional columns, zero row, or zero column. This implies $\det\left(B_{\overrightarrow{\Delta}}\right) = \prod_{i=1}^{n} \lambda_i = 0$. Thus, $0 \in \sigma\left(B_{\overrightarrow{\Delta}}\right)$. □

Recall that $B_{\overrightarrow{\Delta}} = A_{\overline{\overrightarrow{\Delta}}}$. We can say that $0 \in \sigma\left(A_{\overline{\overrightarrow{\Delta}}}\right)$ whenever $0 \in \sigma\left(B_{\overrightarrow{\Delta}}\right)$. Thus, we can present Theorem 3.0.1 as follows.

Remark 3.0.1. *Given a digraph $\overrightarrow{\Delta}$ of order η and its complement digraph $\overline{\overrightarrow{\Delta}}$. $0 \in \sigma\left(A_{\overline{\overrightarrow{\Delta}}}\right)$ if and only if at least one of the following requirement is met:*

(a) *There exist $\nu_i, \nu_j \in V_{\overrightarrow{\Delta}}$ such that $N^+(\nu_i) = N^+(\nu_j)$.*

(b) *There exist $\nu_i, \nu_j \in V_{\overrightarrow{\Delta}}$ such that $N^-(\nu_i) = N^-(\nu_j)$.*

(c) *There exist $\nu_i \in V_{\overrightarrow{\Delta}}$ such that $id_{\overrightarrow{\Delta}}(\nu_i) = \eta$.*

(d) *There exist $\nu_i \in V_{\overrightarrow{\Delta}}$ such that $od_{\overrightarrow{\Delta}}(\nu_i) = \eta$.*

By Theorem 3.0.1, we have Corollary 3.0.1 below.

Corollary 3.0.1. *The spectrum of $B_{\overrightarrow{\Omega_\eta}}$ and $B_{\overrightarrow{\Pi_\eta}}$ does not contain zero.*

Proof. Let $\overrightarrow{\Delta}$ be a cycle digraph or a path digraph. Then,

(a) For every $\nu_i, \nu_j \in V_{\overrightarrow{\Delta}}$, $N^+(\nu_i) \neq N^+(\nu_j)$.

(b) For every $\nu_i, \nu_j \in V_{\overrightarrow{\Delta}}$, $N^-(\nu_i) \neq N^-(\nu_j)$.

(c) For every $\nu_i \in V_{\overrightarrow{\Delta}}$, $id_{\overrightarrow{\Delta}}(\nu_i) \neq n$.

(d) For every $\nu_i \in V_{\overrightarrow{\Delta}}$, $od_{\overrightarrow{\Delta}}(\nu_i) \neq n$.

According to Theorem 3.0.1, we have $0 \notin \sigma\left(B_{\overrightarrow{\Omega_\eta}}\right)$ and $0 \notin \sigma\left(B_{\overrightarrow{\Pi_\eta}}\right)$. □

According to Lemma 2.3.1, one of the eigenvalues of $A_{\overline{\overrightarrow{\Delta}}}$ is not zero if and only if $\overline{\overrightarrow{\Delta}}$ is cyclic. Moreover, by Theorem 3.0.1, we see that the necessary and sufficient condition of $0 \notin \sigma\left(A_{\overline{\overrightarrow{\Delta}}}\right)$. Hence, we have Corollary 3.0.2 below.

Corollary 3.0.2. *Given a digraph $\overrightarrow{\Delta}$ of order n and its complement $\overline{\overrightarrow{\Delta}}$. If all the following conditions are satisfied:*

(a) *for every $\nu_i, \nu_j \in V_{\overrightarrow{\Delta}}$, $N^+(\nu_i) \neq N^+(\nu_j)$,*

(b) *for every $\nu_i, \nu_j \in V_{\overrightarrow{\Delta}}$, $N^-(\nu_i) \neq N^+(\nu_j)$,*

(c) *for every $\nu_i \in V_{\overrightarrow{\Delta}}$, $id_{\overrightarrow{\Delta}}(\nu_i) \neq n$,*

(d) *for every* $\nu_i \in V_{\overrightarrow{\Delta}}$, $od_{\overrightarrow{\Delta}}(\nu_i) \neq n$,

then, $\overrightarrow{\Delta}$ *is a cyclic digraph.*

Proof. From Lemma 2.3.1, we can conclude that one of the eigenvalues of $A_{\overrightarrow{\Delta}}$ is not zero if and only if $\overrightarrow{\Delta}$ is cyclic. By Theorem 3.0.1, we know that $0 \notin \sigma\left(A_{\overrightarrow{\Delta}}\right)$ if and only if the following requirement are all met:

(a) for every $\nu_i, \nu_j \in V_{\overrightarrow{\Delta}}$, $N^+(\nu_i) \neq N^+(\nu_j)$,

(b) for every $\nu_i, \nu_j \in V_{\overrightarrow{\Delta}}$, $N^-(\nu_i) \neq N^+(\nu_j)$,

(c) for every $\nu_i \in V_{\overrightarrow{\Delta}}$, $id_{\overrightarrow{\Delta}}(\nu_i) \neq n$,

(d) for every $\nu_i \in V_{\overrightarrow{\Delta}}$, $od_{\overrightarrow{\Delta}}(\nu_i) \neq n$.

Therefore, if all the requirements are met, then $\overrightarrow{\Delta}$ is cyclic. □

From the properties of eigenvalues of the antiadjacency matrix we have, we cannot find every eigenvalue of the antiadjacency matrix of digraphs, generally. However, we can obtain the bounds of $\rho\left(B_{\overrightarrow{\Delta_c}}\right)$ and $\rho\left(B_{\overrightarrow{\Delta_a}}\right)$.

4 Spectral Radius of Digraphs and Its Complement

This section shows our investigation of $\rho\left(B_{\overrightarrow{\Delta_c}}\right)$ and $\rho\left(B_{\overrightarrow{\Delta_a}}\right)$. First, we discuss the eigenvalues of $B_{\overrightarrow{\Delta_c}}$.

4.1 Cyclic digraphs

If $\overrightarrow{\Delta}$ contains an induced subdigraph that is $\overrightarrow{\Omega_\eta}$, $B_{\overrightarrow{\Delta}}$ includes a principal submatrix that corresponds to $B_{\overrightarrow{\Omega_\eta}}$. According to Theorem 2.2.2, to determine the bounds of $\rho\left(B_{\overrightarrow{\Delta_c}}\right)$, we need to find $\rho\left(B_{\overrightarrow{\Omega_n}}\right)$. Hence, we present the following lemma which states the eigenvalues of $B_{\overrightarrow{\Omega_n}}$.

Lemma 4.1.1. *The eigenvalues of* $B(\overrightarrow{\Omega_\eta})$ *are* $\eta - 1$; *and* $\exp\left(i\pi\left(\frac{(2\zeta+1)(\eta+2)}{\eta}\right)\right)$ *for some natural number* ζ *such that* $\exp\left(i\pi\left(\frac{(2\zeta+1)(\eta+2)}{\eta}\right)\right) \neq -1$.

Proof. Using the principal minors methods, we know that the characteristic polynomial of $B_{\overrightarrow{\Omega_\eta}}$ is given by,

$$P\left(\lambda; B_{\overrightarrow{\Omega_\eta}}\right) = \lambda^\eta + \sum_{\kappa=1}^{\eta}(-1)^\kappa b_\kappa \lambda^{(\eta-\kappa)},$$

where b_κ is the sum of all $\kappa \times \kappa$ principal minor of $B_{\overrightarrow{\Omega_\eta}}$. For $1 \leq \kappa \leq \eta$, all the $\kappa \times \kappa$ principal minors in $B_{\overrightarrow{\Omega_\eta}}$ are the determinant of the antiadjacency matrices of induced subdigraphs of $\overrightarrow{\Omega_\eta}$ of order κ.

All the induced subdigraphs of $\overrightarrow{\Omega_\eta}$ of order $1 \leq \kappa \leq \eta - 1$ are acyclic digraphs. Then, according to Theorem 2.3.2, the determinant of $\kappa \times \kappa$ antiadjacency matrices of induced subdigraphs of $\overrightarrow{\Omega_\eta}$ of order κ is equal to 1 if the induced subdigraphs have a Hamiltonian path, and equal to 0 elsewhere. Since the number of induced subdigraphs of order $\kappa \leq \eta$ that has Hamiltonian path, is η, then $b_\kappa = \eta$ for $1 \leq \kappa \leq \eta - 1$.

Since $\overrightarrow{\Omega_\eta}$ itself is the only subdigraph of order η and according to Theorem 2.3.3 we have the determinant of antiadjacency matrix of subdigraphs of order η is equal to $\eta - 1$, then we have $b_\eta = \eta - 1$.

Consequently, we have the characteristic equation of $B_{\overrightarrow{\Omega_\eta}}$ as follows:

$$\lambda^\eta + \sum_{\kappa=1}^{\eta-1} (-1)^\kappa \eta \lambda^{(\eta-\kappa)} + (-1)^\eta (\eta - 1) = 0.$$

We can use the Horner method to obtain the following characteristic equation:

$$(\lambda - (\eta - 1)) \left(\sum_{\kappa=1}^{\eta-1} (-1)^{\kappa+1} \lambda^{\eta-(\kappa+1)} \right) = 0.$$

Note that $\sum_{\kappa=1}^{\eta} \lambda^{\eta-\kappa} \cdot (-1)^{\kappa+1}$ is an η-geometric series with the first term being $\lambda^{\eta-1}$ and the ratio $-\frac{1}{\lambda}$ which implies the following:

$$(\lambda - (\eta - 1)) \left(\frac{\lambda^\eta + (-1)^{\eta+1}}{\lambda + 1} \right) = 0.$$

Hence, we can conclude that the eigenvalues of $B_{\overrightarrow{\Omega_\eta}}$ are $\eta - 1$ and $\exp\left(i\pi \left(\frac{(2\zeta+1)(\eta+2)}{\eta}\right)\right)$ for some natural number ζ such that $\exp\left(i\pi \left(\frac{(2\zeta+1)(\eta+2)}{\eta}\right)\right) \neq -1$. □

Since $\overrightarrow{\Delta_c}$ has some subdigraphs, which are $\overrightarrow{\Omega_\eta}$, $B_{\overrightarrow{\Delta_c}}$ has a submatrix which is the antiadjacency matrix of a cycle digraph. Thus, from Theorem 2.2.2, to find the bounds of $\rho\left(B_{\overrightarrow{\Delta_c}}\right)$, we need to find $\rho\left(B_{\overrightarrow{\Omega_\eta}}\right)$. From Lemma 4.1.1, we know the eigenvalues of $B_{\overrightarrow{\Omega_\eta}}$. This implies that the following corollary states the value of $\rho\left(B_{\overrightarrow{\Omega_\eta}}\right)$.

Corollary 4.1.1. *Given a cycle digraph $\overrightarrow{\Omega_\eta}$. Then we have,*

$$\rho\left(B_{\overrightarrow{\Omega_\eta}}\right) = \eta - 1. \tag{5}$$

Proof. According to Lemma 4.1.1, the eigenvalues of $B_{\overrightarrow{\Omega_\eta}}$ are $(\eta - 1)$ and $\exp\left(i\pi\left(\frac{(2\zeta+1)(\eta+2)}{\eta}\right)\right)$ for some natural number ζ such that $\exp\left(i\pi\left(\frac{(2\zeta+1)(\eta+2)}{\eta}\right)\right) \neq -1$. Since $max\{\eta-1, |\exp\left(i\pi\left(\frac{(2\zeta+1)(\eta+2)}{\eta}\right)\right)|\} = \eta - 1$, then we have $\rho\left(B_{\overrightarrow{\Omega_\eta}}\right) = \eta - 1$. □

Based on Corollary 4.1.1, we can say that $\overline{\overrightarrow{\Omega_\eta}}$ is $\eta - 1$. Therefore, we can present Corollary 4.1.1 as follows.

Remark 4.1.1. *Given a cycle digraph $\overrightarrow{\Omega_\eta}$ and its complement $\overline{\overrightarrow{\Omega_\eta}}$. Then we have the following:*

$$\rho\left(A_{\overline{\overrightarrow{\Omega_\eta}}}\right) = \eta - 1. \tag{6}$$

After knowing the value of $\rho\left(B_{\overrightarrow{\Omega_\eta}}\right)$ from Corollary 4.1.1, furthermore, motivated by Theorem 2.2.2, we know that the lower bound of $\rho\left(B_{\overrightarrow{\Delta_c}}\right)$ can be determined by finding the spectral radius of the antiadjacency matrix of its subdigraphs which is a cycle digraph. The lower bound is stated in the following theorem below.

Theorem 4.1.1. *Given a digraph $\overrightarrow{\Delta_c}$ that contains a cycle digraph as an induced subdigraph. Then,*

$$\rho\left(B_{\overrightarrow{\Delta_c}}\right) \geq \eta - 1, \tag{7}$$

where η is equal to the order of the largest directed cycle in $\overrightarrow{\Delta_c}$.

Proof. Let $\overrightarrow{\Omega_\mu}$ be an induced subdigraph of $\overrightarrow{\Delta_c}$, where the order of $\overrightarrow{\Omega_\mu}$ is μ. Moreover, suppose that $\Psi_{\overrightarrow{\Omega_\mu}}$, is the antiadjacency matrix of $\overrightarrow{\Omega_\mu}$. Then $\Psi_{\overrightarrow{\Omega_\mu}}$ is a principal submatrix of $B_{\overrightarrow{\Delta_c}}$.

According to Theorem 2.2.2, we have,

$$\rho\left(B_{\overrightarrow{\Delta_c}}\right) \geq \rho\left(\Psi_{\overrightarrow{\Omega_\mu}}\right).$$

By Corollary 4.1.1,

$$\rho\left(\Psi_{\overrightarrow{\Omega_\mu}}\right) = \mu - 1.$$

Therefore, if there are ≥ 1 cycle subdigraphs of $\overrightarrow{\Delta_c}$ and the largest cycle subdigraphs of $\overrightarrow{\Delta_c}$ is of order η, then,

$$\rho\left(B_{\overrightarrow{\Delta_c}}\right) \geq \eta - 1.$$

□

After knowing the lower bound of $\rho\left(B_{\overrightarrow{\Delta_c}}\right)$, we would like to investigate the lower bound of $\rho\left(A_{\overline{\overrightarrow{\Delta_c}}}\right)$. Thus, we have the following corollary.

Corollary 4.1.2. *Given a digraph $\overrightarrow{\Delta_c}$ that contains a cycle digraph as an induced subdigraph, and its complement $\overline{\overrightarrow{\Delta_c}}$. Then, the lower bound of $\rho\left(A_{\overline{\overrightarrow{\Delta_c}}}\right)$ is as follows:*

$$\rho\left(A_{\overline{\overrightarrow{\Delta_c}}}\right) \geq \eta - 1, \tag{8}$$

where η is equal to the order of the largest directed cycle in $\overrightarrow{\Delta_c}$.

Proof. This corollary follows directly from Theorem 4.1.1. □

According to Theorem 2.2.3, the upper bound of $B_{\overrightarrow{\Delta_c}}$ is equal to the maximum sum of all the entries in any row of $B_{\overrightarrow{\Delta_c}}$. Therefore, we obtain the following theorem that states the upper bound of $\rho\left(B_{\overrightarrow{\Delta_c}}\right)$.

Theorem 4.1.2. *Given a cyclic digraph $\overrightarrow{\Delta_c}$ of order n. Then, the upper bound of $\rho\left(B_{\overrightarrow{\Delta_c}}\right)$ is as follows:*

$$\rho\left(B_{\overrightarrow{\Delta_c}}\right) \leq \max\left\{ \left| \left(N^+(\nu_i)\right)^C \right| : \nu_i \in V_{\overrightarrow{\Delta_c}} \right\}. \tag{9}$$

Proof. Let s_i be the sum of the i-th row of $B_{\overrightarrow{\Delta_c}}$ for $i \in \{1, 2, \cdots, n\}$. According to the definition of the antiadjacency matrix, we have,

$$s_i = \sum_{j=1}^{n} b_{i,j} = n - od_{\overrightarrow{\Delta_c}}(\nu_i),$$

for $\nu_i \in V_{\overrightarrow{\Delta_c}}$ and $i \in \{1, 2, ..., n\}$. Therefore, we have,

$$\max\{s_i | i \in \{1, \cdots, n\}\} = n - \min\left\{ od_{\overrightarrow{\Delta_c}}(\nu_i) \ : \nu_i \in V_{\overrightarrow{\Delta_c}} \right\}.$$

According to Theorem 2.2.3, we have,

$$\rho\left(B_{\overrightarrow{\Delta_c}}\right) \leq n - \min\left\{ od_{\overrightarrow{\Delta_c}}(\nu_i) \ : \nu_i \in V_{\overrightarrow{\Delta_c}} \right\}.$$

According to the definition of out-degree and out-neighbourhood, for every vertex $\nu_i \in V_{\overrightarrow{\Delta_c}}$ we have,

$$n - od_{\overrightarrow{\Delta_c}}(\nu_i) = \left| \left(N^+(\nu_i)\right)^C \right|.$$

Therefore, we have the following:

$$n - \min\left\{ od_{\overrightarrow{\Delta_c}}(\nu_i) \ : \nu_i \in V_{\overrightarrow{\Delta_c}} \right\} = \max\left\{ \left| \left(N^+(\nu_i)\right)^C \right| : \nu_i \in V_{\overrightarrow{\Delta_c}} \right\}.$$

Hence, the upper bound of the spectral radius of the antiadjacency matrix of $B_{\overrightarrow{\Delta_c}}$ is the following:

$$\rho\left(B_{\overrightarrow{\Delta_c}}\right) \leq \max\left\{ \left| \left(N^+(\nu_i)\right)^C \right| : \nu_i \in V_{\overrightarrow{\Delta_c}} \right\}.$$

□

Recall that $B_{\overrightarrow{\Delta}} = A_{\overline{\overrightarrow{\Delta}}}$. Thus, from Theorem 4.1.2, we will obtain the following corollary.

Corollary 4.1.3. *Given a cyclic digraph* $\overrightarrow{\Delta_c}$ *and its complement* $\overline{\overrightarrow{\Delta_c}}$. *Then, the upper bound of* $\rho\left(A_{\overline{\overrightarrow{\Delta_c}}}\right)$ *is as follows:*

$$\rho\left(A_{\overline{\overrightarrow{\Delta_c}}}\right) \leq \max\left\{\left|\ \left(N^+(\nu_i)\right)^C\ \right| : \nu_i \in V_{\overrightarrow{\Delta_c}}\right\}. \tag{10}$$

Proof. This corollary follows directly from Theorem 4.1.2. □

Given an arbitrary cyclic digraph $\overrightarrow{\Delta_c}$. Based on our results earlier, we can only generalise for the upper bound of $\rho\left(B_{\overrightarrow{\Delta_c}}\right)$. The upper bound of $\rho\left(B_{\overrightarrow{\Delta_c}}\right)$ is as follows:

$$\rho\left(B_{\overrightarrow{\Delta_c}}\right) \leq \max\left\{\left|\ \left(N^+(\nu_i)\right)^C\ \right| : \nu_i \in V_{\overrightarrow{\Delta_c}}\right\}.$$

Given a digraph $\overrightarrow{\Delta_c}$ that contains a cycle digraph as an induced subdigraph. Based on our earlier results, we can determine the bounds of $\rho\left(B_{\overrightarrow{\Delta_c}}\right)$ are as follows:

$$\eta - 1 \leq \rho\left(B_{\overrightarrow{\Delta_c}}\right) \leq \max\left\{\left|\ \left(N^+(\nu_i)\right)^C\ \right| : \nu_i \in V_{\overrightarrow{\Delta_c}}\right\},$$

where η is the order of the largest cycle in $\overrightarrow{\Delta_c}$. Furthermore, we also establish the following:

$$\eta - 1 \leq \rho\left(A_{\overline{\overrightarrow{\Delta_c}}}\right) \leq \max\left\{\left|\ \left(N^+(\nu_i)\right)^C\ \right| : \nu_i \in V_{\overrightarrow{\Delta_c}}\right\}.$$

Given an example of a cyclic digraph $\overrightarrow{\Delta_{C_1}}$ and its complement $\overline{\overrightarrow{\Delta_{C_1}}}$, depicted in Figure 5 below.

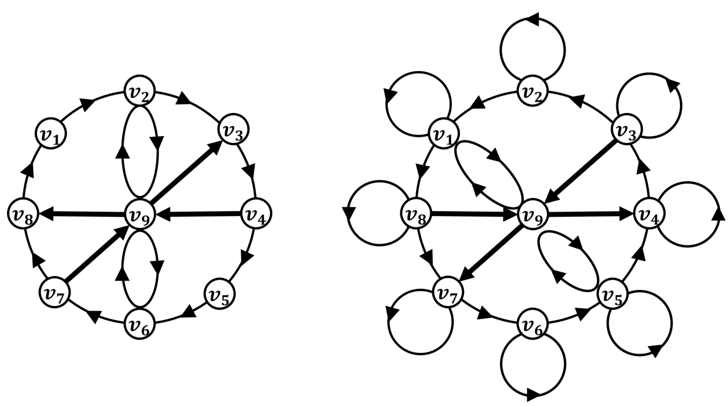

Figure 5: A cyclic digraph $\overrightarrow{\Delta_{C_1}}$ and its complement $\overline{\overrightarrow{\Delta_{C_1}}}$.

We find that the digraph $\overrightarrow{\Delta_{C_1}}$ contains a cycle digraph $\overrightarrow{\Omega_8}$ as an induced subdigraph. Furthermore, $\max\left\{\left|\ \left(N^+(\nu_i)\right)^C\ \right| : \nu_i \in V_{\overrightarrow{\Delta_{C_1}}}\right\} = 8$. Hence, we obtain the bounds of $\rho\left(B_{\overrightarrow{\Delta_{C_1}}}\right)$ as follows:

$$7 \leq \rho\left(B_{\overrightarrow{\Delta_{C_1}}}\right) \leq 8.$$

Moreover, we acquire the bounds of $\rho\left(A_{\overrightarrow{\Delta_{C_1}}}\right)$ as follows:

$$7 \leq \rho\left(A_{\overrightarrow{\Delta_{C_1}}}\right) \leq 8.$$

After determining the bounds of $\rho\left(B_{\overrightarrow{\Delta_c}}\right)$ and $\rho\left(A_{\overrightarrow{\Delta_c}}\right)$, we proceed to examine the bounds of $\rho\left(B_{\overrightarrow{\Delta_a}}\right)$ and $\rho\left(A_{\overrightarrow{\Delta_a}}\right)$.

4.2 Acyclic digraphs

We aim to determine whether the spectral radius of the antiadjacency matrix for acyclic digraphs can be found using a method similar to that used for cyclic digraphs. This section explores the bounds of $\rho\left(B_{\overrightarrow{\Delta_a}}\right)$ and $\rho\left(A_{\overrightarrow{\Delta_a}}\right)$. Since $\overrightarrow{\Pi_\eta}$ is one of the subdigraphs of $\overrightarrow{\Delta_a}$, it is essential to study $B_{\overrightarrow{\Pi_\eta}}$. The following theorem presents the characteristic polynomial of $B_{\overrightarrow{\Pi_\eta}}$.

Lemma 4.2.1. *Given a path digraph $\overrightarrow{\Pi_\eta}$. The characteristic polynomial of $B_{\overrightarrow{\Pi_\eta}}$ is as follows:*

$$P\left(\lambda; B_{\overrightarrow{\Pi_\eta}}\right) = \lambda^\eta + \sum_{\kappa=1}^{\eta}(-1)^\kappa(\eta-(\kappa-1))\lambda^{(\eta-\kappa)}. \tag{11}$$

Proof. Using the principal minors method, we know that the characteristic polynomial of $B_{\overrightarrow{\Pi_\eta}}$ is given by,

$$P\left(\lambda; B_{\overrightarrow{\Pi_\eta}}\right) = \lambda^\eta + \sum_{\kappa=1}^{\eta}(-1)^\kappa c_\kappa\lambda^{(\eta-\kappa)},$$

where c_κ represents the sum of all $\kappa\times\kappa$ principal minor of $B_{\overrightarrow{\Pi_\eta}}$. For $1 \leq \kappa \leq \eta$, these $\kappa\times\kappa$ principal minors of $B_{\overrightarrow{\Pi_\eta}}$ are the determinant of the antiadjacency matrices of the induced subdigraphs of $\overrightarrow{\Pi_\eta}$ with order κ.

Every induced subdigraphs of $\overrightarrow{\Pi_\eta}$ of order $1 \leq \kappa \leq \eta$ are acyclic digraphs. Then, by Theorem 2.3.2, the determinant of $\kappa \times \kappa$ antiadjacency matrices of induced subdigraphs of $\overrightarrow{\Pi_\eta}$ of size κ is equal to 1 if the subdigraphs have a Hamiltonian path, and equal to 0 elsewhere. Since the number of every induced subdigraph of $\overrightarrow{\Pi_\eta}$ of order $\kappa \leq \eta$, that are Hamiltonian path, is $\eta-(\kappa-1)$, then $b_\kappa = (-1)^\kappa \cdot (\eta-(\kappa-1))$ for $1 \leq \kappa \leq \eta-1$.

Hence, we can conclude the characteristic polynomial of $B_{\overrightarrow{\Pi_n}}$ as follows:

$$P\left(\lambda; B_{\overrightarrow{\Pi_n}}\right) = \lambda^\eta + \sum_{\kappa=1}^{\eta}(-1)^\kappa(\eta-(\kappa-1))\lambda^{(\eta-\kappa)}.$$

□

According to Theorem 2.2.1 and Lemma 4.2.1, $P\left(\lambda; B_{\overrightarrow{\Pi}_\eta}\right)$ can be represented as an arithmetic-geometric series. Finding the roots of the characteristic polynomial of $B_{\overrightarrow{\Pi}_\eta}$ is equivalent to determining its eigenvalues. Thus, we are still having a difficulty on determining the eigenvalues of $B_{\overrightarrow{\Pi}_n}$. Therefore, we leave it as the following open problem.

Open Problem 4.2.1. *Given a path digraph $\overrightarrow{\Pi}_\eta$ and its antiadjacency matrix $B_{\overrightarrow{\Pi}_\eta}$. What are the eigenvalues of $B_{\overrightarrow{\Pi}_\eta}$?*

In addition to Open Problem 4.2.1, we have difficulties in determining the value of $\rho\left(B_{\overrightarrow{\Pi}_\eta}\right)$. Hence, we leave it as an open problem in the following.

Open Problem 4.2.2. *Given a path digraph $\overrightarrow{\Pi}_\eta$ and its antiadjacency matrix $B_{\overrightarrow{\Pi}_\eta}$. What is the value of $\rho\left(B_{\overrightarrow{\Pi}_\eta}\right)$?*

Although we have an Open Problem 4.2.2 on obtaining the spectral radius of $B_{\overrightarrow{\Pi}_\eta}$, we can estimate its spectral radius using Theorem 2.2.3. Theorem 4.2.1 provides the bounds of $\rho\left(B_{\overrightarrow{\Pi}_\eta}\right)$.

Theorem 4.2.1. *Given a path digraph $\overrightarrow{\Pi}_\eta$. Then, the bounds of $\rho\left(B_{\overrightarrow{\Pi}_\eta}\right)$ are as follows:*

$$\eta - 1 \le \rho\left(B_{\overrightarrow{\Pi}_\eta}\right) \le \eta. \tag{12}$$

Proof. Let s_i represent the sum of the i-th row of $B_{\overrightarrow{\Pi}_\eta}$. The definition of the antiadjacency matrix gives us:

$$s_i = \eta - od_{\overrightarrow{\Pi}_\eta}(\nu_i),$$

for $\nu_i \in V_{\overrightarrow{\Pi}_\eta}$ and $i \le \eta$. Thus, we have,

$$\min\left\{od_{\overrightarrow{\Pi}_\eta}(\nu_i) \,:\, \nu_i \in V_{\overrightarrow{\Pi}_\eta}\right\} = 0,$$

and

$$\max\left\{od_{\overrightarrow{\Pi}_\eta}(\nu_i) \,:\, \nu_i \in V_{\overrightarrow{\Pi}_\eta}\right\} = 1.$$

Therefore, we have,

$$\min\{s_i | i \in \{1, 2, ..., \eta\}\} = \eta - 1,$$

and

$$\max\{s_i | i \in \{1, 2, ..., \eta\}\} = \eta.$$

According to Theorem 2.2.3 we conclude that,

$$\eta - 1 \le \rho\left(B_{\overrightarrow{\Pi}_\eta}\right) \le \eta.$$

□

From Theorem 4.2.1, we obtain the bounds of $A_{\overrightarrow{\overline{\Pi}_n}}$.

Corollary 4.2.1. *Given a path digraph* $\overrightarrow{\Pi}_\eta$ *and its complement* $\overrightarrow{\overline{\Pi}}_\eta$. *Then, the bounds of* $\rho\left(A_{\overrightarrow{\overline{\Pi}_n}}\right)$ *are as follows:*

$$\eta - 1 \leq \rho\left(A_{\overrightarrow{\overline{\Pi}_n}}\right) \leq \eta. \tag{13}$$

Proof. This corollary is the direct implication of Theorem 4.2.1. □

We cannot use Theorem 4.1.1 to obtain the lower bound of $\rho\left(B_{\overrightarrow{\Delta_a}}\right)$ as the value of $\rho\left(B_{\overrightarrow{\Pi_\eta}}\right)$ cannot be found. As a result, we must use another approach to determine its lower bound.

Theorem 4.2.1 provides information on spectral radius of antiadjacency matrix for a path digraph, which itself is an acyclic digraph. As a result, in this part, we generalise the spectral radius of antiadjacency matrix for all acyclic digraphs. Theorem 4.2.2 shows the bounds of $\rho\left(B_{\overrightarrow{\Delta_a}}\right)$.

Theorem 4.2.2. *Given an acyclic digraph* $\overrightarrow{\Delta_a}$ *of order* η. *Then, the bounds of* $\rho\left(B_{\overrightarrow{\Delta_a}}\right)$ *are as follows:*

$$\min\left\{|\left(N^+(\nu_i)\right)^C| : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\} \leq \rho\left(B_{\overrightarrow{\Delta_a}}\right) \leq \max\left\{|\left(N^+(\nu_i)\right)^C| : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\}. \tag{14}$$

Proof. Suppose that s_i represents the summation of the i-th row of $B_{\overrightarrow{\Delta_a}}$ for $i \leq \eta$. Based on the definition of the antiadjacency matrix, we have,

$$s_i = \sum_{j=1}^{\eta} b_{i,j} = \eta - od_{\overrightarrow{\Delta_a}}(\nu_i),$$

for every vertex $\nu_i \in V_{\overrightarrow{\Delta_a}}$.

Therefore, we have,

$$\min\{s_i : i \in \{1, \cdots, \eta\}\} = \eta - \max\left\{od_{\overrightarrow{\Delta_a}}(\nu_i) : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\},$$

and

$$\max\{s_i : i \in \{1, \cdots, \eta\}\} = \eta - \min\left\{od_{\overrightarrow{\Delta_a}}(\nu_i) : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\}.$$

Thus, Theorem 2.2.3 gives us:

$$\eta - \max\left\{od_{\overrightarrow{\Delta_a}}(\nu_i) : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\} \leq \rho\left(B_{\overrightarrow{\Delta_a}}\right) \leq \eta - \min\left\{od_{\overrightarrow{\Delta_a}}(\nu_i) : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\}.$$

According to the definition of out-degree and out-neighbourhood, we have $\eta - od_{\overrightarrow{\Delta_a}}(\nu_i) = |\left(N^+(\nu_i)\right)^C|$ for every vertex $\nu_i \in V_{\overrightarrow{\Delta_a}}$. Therefore, we have,

$$\eta - \max\left\{od_{\overrightarrow{\Delta_a}}(\nu_i) : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\} = \min\left\{|\left(N^+(\nu_i)\right)^C| : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\},$$

and

$$\eta - \min\left\{od_{\overrightarrow{\Delta_a}}(\nu_i) : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\} = \max\left\{\mid \left(N^+(\nu_i)\right)^C \mid : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\}.$$

Hence, we have the bounds of $\rho\left(B_{\overrightarrow{\Delta_a}}\right)$ as follows:

$$\min\left\{\left|\left(N^+(\nu_i)\right)^C\right| : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\} \leq \rho\left(B_{\overrightarrow{\Delta_a}}\right) \leq \max\left\{\left|\left(N^+(\nu_i)\right)^C\right| : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\}.$$

□

We have the following corollary as a consequence of the definition of the antiadjacency matrix of a digraph.

Corollary 4.2.2. *Given an acyclic digraph $\overrightarrow{\Delta_a}$ of order η and its complement $\overline{\overrightarrow{\Delta_a}}$. Then, we have the following:*

$$\min\left\{\left|\left(N^+(\nu_i)\right)^C\right| : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\} \leq \rho\left(A_{\overline{\overrightarrow{\Delta_a}}}\right) \leq \max\left\{\left|\left(N^+(\nu_i)\right)^C\right| : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\}. \tag{15}$$

Proof. This corollary is the direct implication of Theorem 4.2.2. □

Based on our earlier results, we obtain the bounds of $\rho\left(B_{\overrightarrow{\Delta_a}}\right)$ as follows:

$$\min\left\{\left|\left(N^+(\nu_i)\right)^C\right| : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\} \leq \rho\left(B_{\overrightarrow{\Delta_a}}\right) \leq \max\left\{\left|\left(N^+(\nu_i)\right)^C\right| : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\}.$$

We also obtain the bounds of $\rho\left(A_{\overline{\overrightarrow{\Delta_a}}}\right)$ is the following:

$$\min\left\{\left|\left(N^+(\nu_i)\right)^C\right| : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\} \leq \rho\left(A_{\overline{\overrightarrow{\Delta_a}}}\right) \leq \max\left\{\left|\left(N^+(\nu_i)\right)^C\right| : \nu_i \in V_{\overrightarrow{\Delta_a}}\right\}.$$

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